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On the Divergence of Spectral Expansions of Elliptic Differential Operators

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ABSTRACT

In this paper we consider spectral expansions of functions from Nikol'skii classes $H_p^a(R^N)$, related to selfadjoint extensions of elliptic differential operators $A(D)$ of order m in R^N. We construct a continuous function from Nikol'skii class with $pa < N$, such that the Riesz means of spectral expansion of which diverge at the origin. This result demonstrates sharpness of the condition $pa > N$ obtained earlier by Alimov (1976) for uniform convergence of spectral expansions, related to elliptic differential operators.

Keywords: Fourier integral, spectral expansions of the differential operators, spectral function, Riesz means.

INTRODUCTION

Let G be an arbitrary domain in R^N. In $L_2(G)$ consider a formally self-adjoint and semi-bounded elliptic differential operator $A(x,D)$ with smooth coefficients. For any $f \in L_2(G)$ we denote its Riesz means of order $s \geq 0$ by $E_\lambda^s f(x)$. The aim of this paper is to investigate the problem of uniform convergence of these means. In Alimov (1976) considered the Riesz means of the functions from Nikolskii classes $H_p^a(G)$ (for definition see Nikolskii (1969)). He proved that if numbers $a > 0$ and $s \geq 0$ satisfy the conditions

$$a+s\geq\frac{N-1}{2},\qquad pa>N,\qquad p\geq 1,\tag{1}$$

then for any function $f\in H_p^a(G)$ with compact support the Riesz means $E_\lambda^s f(x)$ converge to the function $f(x)$ uniformly on any compact set $K\subset G.$

For the Laplace operator in the case $s=0$ Il'in obtained the sharp conditions (1) for the uniformly convergence of spectral expansions for functions from Sobolev spaces. Later this result was extended to the Liuoville spaces by Il'in and Moiseev in (Il'in and Moiseev (1971)) and to the Nikolskii spaces by Il'in and Alimov in (Il'in and Alimov (1971)). The problem of the uniformly convergence of the spectral decompositions of the Laplace operator in the critical case $pa=N$ is investigated by Alimov in (Alimov (1979)), where he proved uniformly convergence of the Riesz means $E_\lambda^s f(x)$ for the continuous function $f(x)$ from Sobolev space $W_p^a(G)$ under the conditions $a+s>(N-1)/2,\quad p\geq 1.$

The critical case $pa=N$ was investigated in (Alimov (1979)) for the spectral decompositions of the Laplace operator, and the uniformly convergence of the Riesz means $E_\lambda^s f(x)$ is established when the being expanded function is continuous and belongs to the Sobolev space $W_p^a(G)$ and the first inequality in (1) is strong. In this connection there arises a natural question: If we replace the condition $pa\geq N$ with $pa<N,$ and request that the function $f\in H_p^a(G)$ is continuous, then whether or not the uniformly convergence of the Riesz means of the spectral decomposition of the general elliptic operator remains valid (possibly at the expense of increasing the $a+s$). In this paper we give a negative answer to this question.

PROBLEM FORMULATION

For a semibounded elliptic operator $A(D)$ with constant coefficients in case of $G=R^N$ the decomposition unity is defined by

$$E_\lambda f(x) = (2\pi)^{-N} \int\limits_{A(\xi)<\lambda} \hat{f}(\xi) e^{i(x,\xi)} d\xi,$$

where $A(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha$ is the symbol of the operator $A(D)$, and $\hat{f}$ denotes, Fourier transformation of f. Then the Riesz means of order s of the E_λ is defined by

$$E_\lambda^s f(x) = \int_{\lambda_0}^{\lambda} (1-\frac{t}{\lambda})^s dE_t f(x), \tag{2}$$

where λ_0 is lower bound of the $A(D)$.

The main result of this paper is the following

Theorem 1. *Let* $0 \le s \le \frac{N-1}{2}$, $p \ge 1$, $ap < N$ *(here* $a+s$ *is arbitrary). Then there exists a continuous function* $f(x)$ *from the class* $H_p^a(R^N)$ *such that*

$$\overline{\lim_{\lambda\to\infty}} | E_\lambda^s f(0) | = +\infty. \tag{3}$$

Thus Theorem in particular states that if $ap < N$, then even the inequality $a+s > (N-1)/2$ can not guarantee the uniformly convergence of the spectral expansions of the elliptic differential operators. It should be noted that using an explicit form for the kernel of $E_\lambda^s f$ via Bessel function, in case of the Laplace operator this theorem was proved by the first author (Ashurov (1990)). The operators $A(D)$ considered in the present paper have an arbitrary order and are not necessarily homogenous. Moreover, in our case the surface $\{a(\xi)=1\}$ can be not convex, where $a(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ denotes the principle symbol of the operator $A(D)$. Whereas for the Laplace operator it is a sphere and therefore it is a strictly convex set. So in the case of an arbitrary elliptic operator we have the same result as for Laplace operator.

For the latest investigations related to the convergence problems of the spectral expansions of the elliptic differential operators (including the case of the Laplace operator) we refer the readers to (Ashurov and Anvarjon (2010); Ashurov *et al.* (2010); Anvarjon (2010); Carbery and Soria (1988); Tao (2002); Alimov (2006); Carbery and Soria (1997)). The best general reference here is Alimov *et al.* (1992).

RESULTS AND DISCUSSION

Let us recall the Nikol'skii class of functions, which is denoted by $H_p^a(R^N)$, where $a = l + \kappa$, l-positive integer and $0 < \kappa \le 1$, $p \ge 1$. We say that the function $f \in L_p(R^N)$, $p \ge 1$, belongs to the $H_p^a(R^N)$, if for any $h \in R^N$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$, $|\alpha| = l$, we have

$$\| \Delta_h^2 \partial^\alpha f(x) \|_{L_p(R^N)} \le c\,|h|^\kappa,$$

where $\partial^\alpha = \dfrac{\partial^{\alpha_1+\alpha_2+...+\alpha_N}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_N^{\alpha_N}}$ and $\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h).$

The norm in $H_p^a(R^N)$ is defined by

$$\| f \|_{p,a} = \| f \|_{L_p(R^N)} + \sum_{|\alpha|=l} \sup_h |h|^{-\kappa} \| \Delta_h^2 \partial^\alpha f(x) \|_{L_p(R^N)},$$

If $\kappa < 1$, then the second order difference in this representation can be replaced with the first order difference $\Delta_h \partial^\alpha f(x) = \partial^\alpha f(x+h) - \partial^\alpha f(x)$, for more properties see (Nikolskii (1969)).

Let Ω be a domain on the unit sphere. We define a set

$$K_\Omega(x, b_1, b_2) = \{ y : b_1 < |x-y| < b_2, (x-y)/|x-y| \in \Omega \}.$$

Let $\psi(\omega), \varphi(\omega)$ and $\phi(\omega)$ be smooth functions defined on unit sphere S^{N-1}.

Let $\mu = \lambda^{1/m} > 1,\ \varphi_0 = \frac{\pi}{2}(\frac{N-1}{2} - s),$ then we define

$$b_1 = (\ln\ln\mu + \beta_1 - \varphi_0)(\mu\psi(\omega) + \phi(\omega))^{-1},$$

$$b_2 = (ln\mu + b_0)(\mu\psi(\omega) + \phi(\omega))^{-1},$$

where $\beta_1 = \beta_1(\mu)$ and $\beta_2 = \beta_2(\mu)$ are numbers such that

$$cos(\ln\ln\mu + \beta_1(\mu)) = 1 \text{ and } cos(\ln\mu + \beta_2(\mu) + \varphi_0) = 1.$$

It is further assumed that $0 \le \beta_1, \beta_2 < 2\pi.$ We introduce the function

$$g_\mu(t,\omega) = \begin{cases} cos(\mu t\psi(\omega) + t\phi(\omega) + \varphi_0), & if \quad t \in (b_1, b_2), \omega \in \Omega \\ 0 \quad , & if \quad t \bar{\in} (b_1 - \pi/(2\mu), b_2 + \pi/(2\mu)). \end{cases}$$

We extend $g_\mu(t,\omega)$ smoothly to the rest of the $K_\Omega(0, b_1 - \frac{\pi}{2\mu}, b_2 + \frac{\pi}{2\mu})$, so we have $|\partial^\alpha g_\mu(t,\omega)| \le C\mu^{|\alpha|},\ C > 0,$ uniformly in t and ω for all $0 \le \alpha \le N.$

Let $\chi(\omega)$ be smooth function defined on unit sphere S^{N-1} with $supp(\chi) \subset \Omega.$ We consider a function $f_\mu(y) = g_\mu(|y|,\omega)\chi(\omega)e^{-i|y|\varphi(\omega)}$, where $\omega = y/|y| \in \Omega.$

Lemma 1. *If* $0 \le a < N,$ *then for any* $\mu > 1$ *we have*

$$\| f_\mu \|_{1,a} \le A\mu^{a-N}(ln\mu)^N, \quad A > 0.$$

Proof. In the definition of the H_1^a norm we take difference of first order of $\partial^\alpha f_\mu$. It is not hard to see that

$$\| f_\mu \|_{L_1(R^N)} \le c(\mu^{-1}ln\mu)^N.$$

The second summand in

$$\| f_\mu \|_{1,a} = \| f_\mu \|_{L_1(R^N)} + \sum_{|\alpha|=N-1} \sup_h |h|^{-\kappa} \| \Delta_h \partial^\alpha f_\mu \|_{L_1(R^N)},$$

we divide into two parts as follows

$$\sum_{|\alpha|=N-1} \sup_{|h|\ge \mu^{-1}} |h|^{-\kappa} \| \Delta_h \partial^\alpha f_\mu \|_{L_1(R^N)} + \sum_{|\alpha|=N-1} \sup_{|h|<\mu^{-1}} |h|^{-\kappa} \| \Delta_h \partial^\alpha f_\mu \|_{L_1(R^N)}$$

$$= I_1 + I_2.$$

For I_1 we have:

$$I_1 = \sum_{|\alpha|=N-1} \sup_{|h|\ge \mu^{-1}} |h|^{-\kappa} \| \Delta_h \{ \partial^\alpha g_\mu(|y|,\omega) \} \|_{L_1(R^N)} \le$$

$$\le \sum_{|\alpha|=N-1} \sup_{|h|\ge \mu^{-1}} |h|^{-\kappa} \left(\| \partial^\alpha g_\mu(|y+h|,\omega_h) \|_{L_1(R^N)} + \| \partial^\alpha g_\mu(|y|,\omega) \|_{L_1(R^N)} \right) \le$$

$$\le C \sum_{m=0}^{|\alpha|} \sum_{|\alpha|=N-1} \mu^\kappa \mu^m \int_I t^{N+m-|\alpha|-1} dt \le C \sum_{m=0}^{|\alpha|} \sum_{|\alpha|=N-1} \mu^{\kappa+m} (\mu^{-1} ln\mu)^{N+m-|\alpha|}$$

$$\le C \mu^{a-N} (ln\mu)^N,$$

where $\omega = \frac{y}{|y|}$ and $\omega_h = \frac{y+h}{|y+h|}$. To estimate I_2 we will apply the Leibniz formula:

$$I_2 = \sum_{|\alpha|=N-1} \sup_{|h|<\mu^{-1}} |h|^{-\kappa} \| \Delta_h \sum_{m=0}^{|\alpha|} \sum_{|\beta|=m} C_{\alpha\beta} \partial^\beta (g_\mu(|y|,\omega)) \cdot \partial^{\alpha-\beta} (\chi(\omega) e^{-i|y|\varphi(\omega)}) \|_{L_1(R^N)}.$$

In this case it is convenient to use the identity

$$\Delta_h \left\{ \partial^\beta (g_\mu(y)) \cdot \partial^{\alpha-\beta} (\chi(\omega) e^{-i|y|\varphi(\omega)}) \right\} =$$

$$= \partial^\beta g_\mu(y+h) \Delta_h \{ \partial^{\alpha-\beta} (\chi(\omega) e^{-i|y|\varphi(\omega)}) \} + \partial^{\alpha-\beta} (\chi(\omega) e^{-i|y|\varphi(\omega)}) \Delta_h \{ \partial^\beta g_\mu(y) \}.$$

Since $\psi(\omega),\varphi(\omega),\phi(\omega)$ satisfy conditions $|\Psi(\omega)|\leq C,\ \Delta_h(\Psi)\leq C|h|$, so to prove the inequality in Lemma 1 it suffices to get the following inequality

$$\|\mu^m |y+h|^{m-|\beta|} C_\alpha \Delta_h\{|y|^{k-|\alpha|+|\beta|}\cos(|y|\phi(\omega))\}\|_{L_1}\leq C|h|^\kappa \mu^{a-N}(\ln\mu)^N,$$

for any $m=0,1,2,...|\alpha|, k=0,1,2,...,|\alpha|-|\beta|,|\beta|\leq|\alpha|$.

It is easy to see that $|\Delta_h |y|^{k-|\alpha|+|\beta|}\leq C|y|^{k-|\alpha|+|\beta|-1}|h|$. Hence for any $m=0,1,2,...|\alpha|, k=0,1,2,...|\alpha|-|\beta|,|\beta|\leq|\alpha|$ we obtain

$$\|\mu^m |y+h|^{m-|\beta|} C_\alpha \Delta_h\{|y|^{k-|\alpha|+|\beta|}\cos(|y|\phi(\omega))\}\|_{L_1}\leq$$

$$\leq C|h|\mu^m \int\limits_{|y+h|\in I} |y+h|^{m-|\beta|}|y|^{k-|\alpha|+|\beta|-1}|y|^{N-1}\,d|y|\leq$$

$$\leq C|h|\mu^m(\mu^{-1}\ln\mu)^{m+k+N-|\alpha|}\leq C|h|^\kappa \mu^{a-N}(\ln\mu)^N.$$

Finally for the I_2 we have

$$I_2=\sum_{|\alpha|=N-1}\sup_{|h|<\mu^{-1}}|h|^{-\kappa}\|\Delta_h\sum_{m=0}^{|\alpha|}\sum_{|\beta|=m}C_{\alpha\beta}\partial^\beta(g_\mu(y))\cdot\partial^{\alpha-\beta}(\chi(\omega)e^{-i|y|\varphi(\omega)})\|_{L_1(R^N)}\leq$$

$$\leq C\mu^{a-N}(\ln\mu)^N.$$

So statement of Lemma 1 is proved.

We obtain estimation from below for the Riesz means of the spectral decompositions of the function $f_\mu(x)$. The asymptotic behavior of the spectral function $\Theta(x,y,\lambda)$ and its Riesz means

$$\Theta^s(x,y,\lambda)=\int\limits_{\lambda_0}^{\lambda}\left(1-\frac{t}{\lambda}\right)^s d\Theta(x,y,t),$$

for large values of λ plays an important role in the study of problems of convergence and summability of spectral decompositions. The investigation of the spectral function $\Theta(x,y,\lambda)=(2\pi)^{-N}\int\limits_{A(\xi)<\lambda} e^{i(x-y,\xi)}d\xi,$ is equivalent to that of the Fourier transform of the characteristic function of the set $\{A(\xi)<1\}$. Of course, the behavior of $\Theta(x,y,\lambda)$ depends essentially on the geometry of $\{A(\xi)<1\}$. We say that an operator $A(D)$ has a strictly convex symbol if $a(\xi)$ is the principal symbol of $A(D)$, and all the $N-1$ principal curvatures of the surface $\{a(\xi)=1\}$ are everywhere different from 0. If the surface $\{a(\xi)=1\}$ is convex set, then we say that the elliptic differential operator $A(D)$ has convex symbol. Note that in this case some of principal curvatures of the surface $\{a(\xi)=1\}$ may be equal to zero.

Let Ω be an arbitrary domain on the unit sphere and let E_Ω be the set of points of the surface $\{a(\xi)=1\}$, where the exterior normal vector coincides with $\omega\in\Omega$. We shall say that the point $\xi\in E_\Omega$ is a point of strict convexity if all the principal curvatures at this point are different from zero. We choose the domain Ω so that all the points of the set E_Ω be points of strict convexity. Since the symbol $a(\xi)$ is polynomial, such domain always exists.

Lemma 2. *On the unit sphere there exists a domain* Ω *such that for all* $x,y,$ *for which* $\omega=(x-y)/|x-y|,$ $x\neq y,$ *we have the asymptotic representation for* $0\le s<(N-1)/2$:

$$\Theta^s(x,y,\lambda)=C\lambda^{N/m}e^{i|x-y|\varphi(\omega)}H^{-1/2}(\omega)$$

$$(cos(\lambda^{1/m}|x-y|\psi(\omega)+|x-y|\phi(\omega)+\varphi_0)\psi^{-s}(\omega)(\lambda^{-1/m}|x-y|)^{-(N+1)/2-s}+$$

$$+O(1)(\lambda^{-1/m}|x-y|)^{-(N+3)/2-s}),$$

where $H(\omega)$ is the Gauss curvature of the surface $a(\xi)=1$ at the points of contact with the hyperplane orthogonal to $\omega,$ $\psi(\omega)=\sup\limits_{a(\xi)=1}(\omega,\xi),$ $\varphi(\omega),\phi(\omega)$ are smooth functions depending on the $A(\xi)$, full symbol of the operator.

The proof of this Lemma may be conducted using the method of stationary phase as in the work (Ashurov (1981)), where the same result is proved for elliptic operators with convex symbol. Lemma 2 is generalization of the latter for an arbitrary elliptic differential operator $A(D)$ with constant coefficients.

Theorem 2. *Let* $\lambda > max\{1,\lambda_0\}$ *and* $\mu = \lambda^{1/m}$, *then there exists a constant* $B > 0$ *such that*

$$| E_\lambda^s f_\mu(0) | \geq B max\{ln ln \mu, (ln \mu)^{\frac{N-1}{2}-s}\}, \quad 0 \leq s \leq \frac{N-1}{2}.$$

Proof. Using the asymptotical formula for $\Theta^s(x,y,\lambda)$ and passing to the spherical coordinates we have

$$E_\lambda^s f_\mu(0) =$$
$$c\mu^{\frac{N-1}{2}-s} \int_{b_1-\pi/2\mu}^{b_2+\pi/2\mu} \int_\Omega \frac{e^{it\varphi(\omega)} \cos((z_\mu(\omega)t+\varphi_0) f_\mu(t\omega)}{\sqrt{H(\omega)}\psi^s(\omega) t^{\frac{N+1}{2}+s}} t^{N-1} dt d\omega + O(1)(1+(\ln\mu)^{\frac{N-3}{2}-s}),$$

where $z_\mu(\omega) = \mu\psi(\omega) + \phi(\omega)$. Using Fubini's theorem we obtain (we have known that $\psi(\omega) \geq c_0 > 0, \omega \in \Omega$):

$$E_\lambda^s f_\mu(0) = c\mu^{\frac{N-1}{2}-s} \int_\Omega \int_{b_1-\pi/2\mu}^{b_2+\pi/2\mu} t^{\frac{N-1}{2}-s-1} e^{it\varphi(\omega)} \frac{\cos(tz_\mu(\omega)+\varphi_0) f_\mu(t,\omega)}{\sqrt{H(\omega)}\psi^s(\omega)} dt d\omega +$$

$$O(1)(1+(\ln\mu)^{\frac{N-3}{2}-s}).$$

If we denote the first summand by T, then by definition of the function $f_\mu(y)$ we shall have

$$T \geq c\mu^{\frac{N-1}{2}-s} \int_\Omega \frac{1}{\sqrt{H(\omega)}\psi^s(\omega)} d\omega \int_{b_1}^{b_2} t^{\frac{N-1}{2}-s-1} \cos^2(z_\mu(\omega)t+\varphi_0) dt.$$

Since $\cos^2\gamma = \frac{1}{2} + \frac{1}{2}\cos 2\gamma$ we obtain

$$T \ge \frac{c}{2}\mu^{\frac{N-1}{2}-s} \int_\Omega \int_{b_1}^{b_2} \frac{t^{\frac{N-1}{2}-s-1}}{\sqrt{H(\omega)}\psi^s(\omega)} d\omega dt +$$

$$+\frac{c}{2}\mu^{\frac{N-1}{2}-s} \int_\Omega \int_{b_1}^{b_2} \frac{t^{\frac{N-1}{2}-s-1} \cos 2(z_\mu(\omega)t + \varphi_0)}{\sqrt{H(\omega)}\psi^s(\omega)} d\omega dt = S_1 + S_2.$$

In the interior integral of S_2 integrating by part we have

$$S_2 = \frac{c}{2}\mu^{\frac{N-1}{2}-s} \int_\Omega \frac{1}{\sqrt{H(\omega)}\psi^s(\omega)} d\omega \left\{ \frac{t^{\frac{N-1}{2}-s-1} \sin 2(z_\mu(\omega)t + \varphi_0)}{(\mu\psi(\omega) + \varphi(\omega))} \Big|_{b_1}^{b_2} - \right.$$

$$\left. - \frac{\frac{N-1}{2} - s}{z_\mu(\omega)} \int_{b_1}^{b_2} t^{\frac{N-3}{2}-s-1} \sin 2(z_\mu(\omega)t + \varphi_0) dt \right\}.$$

Using the conditions $cos(\ln\ln\mu + \beta_1(\mu)) = 1,\ cos(\ln\mu + \beta_2(\mu) + \varphi_0) = 1$ we obtain

$$S_2 = C\mu^{\frac{N-3}{2}-s} \int_\Omega \frac{1}{\sqrt{H(\omega)}\psi^s(\omega)} d\omega \int_{b_1}^{b_2} t^{\frac{N-3}{2}-s-1} \sin 2(z_\mu(\omega)t + \varphi_0) dt.$$

From the continuity of the functions $H(\omega)$ and $\psi(\omega)$ on Ω it follows that $c_1 \le H(\omega) \le c_2$, $c_3 \le \psi(\omega) \le c_4$, where $c_j, j = 1,2,3,4$ are positive constants. By virtue of these conditions and of the definition of $K_\Omega(x, b_1, b_2)$, one can assert that

$$|S_2| \le C(1 + (\ln\mu)^{\frac{N-1}{2}-s-1}).$$

So we obtain for T the following estimation from below:

$$T \geq || S_1 | - | S_2 || \geq C\mu^{\frac{N-1}{2}-s} \int_{\Omega}\int_{b_1}^{b_2} t^{\frac{N-1}{2}-s-1} dt + O(1)(1+\ln\mu)^{\frac{N-3}{2}-s}.$$

Finally we have

$$T \geq \begin{cases} c\ln\ln\mu, & s = \dfrac{N-1}{2}, \\ c(\ln\mu)^{\frac{N-1}{2}-s}, & s \neq \dfrac{N-1}{2}. \end{cases}$$

Theorem 2 is proved. Now Theorem 1 can be proved by similar way as in (Ashurov (1990)).

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